Periods of curves without Riemann integrals

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MAX-PLANCK-GESELLSCHAFT

Part I

Introduction

Why periods?

Before even defining periods. . .

Why would anyone be interested in periods?

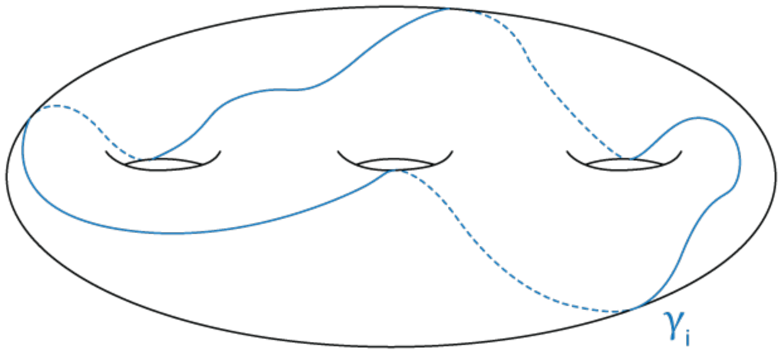
- Soliton equations of KdV hierarchies.
- Geometry of varieties are hidden in their periods.
- Numerical foothold onto Hodge theory.

Higher dimensional story:

1. Sertöz, *Computing periods of hypersurfaces*, `arXiv:1803.08068`
2. Lairez, Sertöz, *Numerical determination of Hodge cycles on hypersurfaces*, in preparation.

What are periods?

Let C be a compact Riemann surface of genus g. Fix a basis for the first homology group $\gamma_1, \ldots, \gamma_{2g} \in \mathrm{H}_1(C, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$.



- We have an intersection product $\gamma_i \cdot \gamma_j \in \mathbb{Z}$.

Fix a basis for the space of *holomorphic* differential forms $\omega_1, \ldots, \omega_g \in \mathrm{H}^0(C, \omega_C) \subset \mathrm{H}^1_{\mathrm{dR}}(C, \mathbb{C})$.

Definition

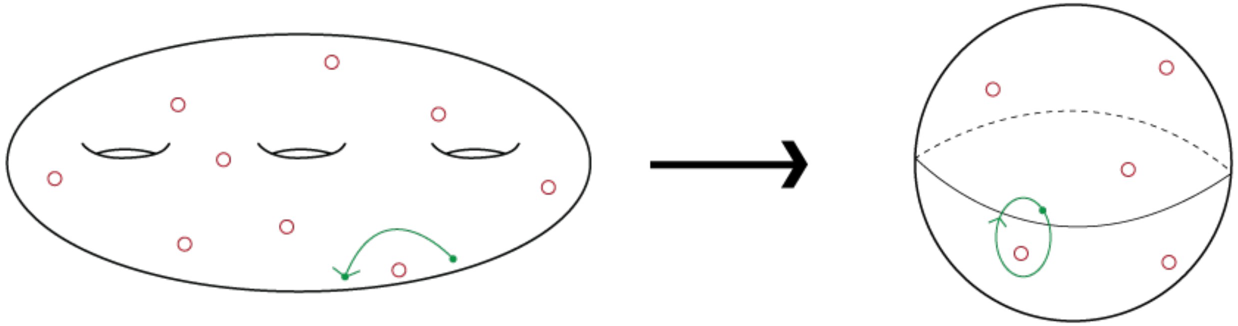
The *period matrix* of C corresponding to the bases above is:

$$\left(\int_{\gamma_j} \omega_i \right)_{\substack{i=1,\ldots,g \\ j=1,\ldots,2g}} \in \mathbb{C}^{g \times 2g}$$

Period computation with Riemann integrals

Take a (possibly singular) planar model of C as the zero set of a polynomial $f(x, y, z)$ in $\mathbb{P}^2_{\mathbb{C}}$.

- Compute holomorphic forms on C using 1-forms on $\mathbb{P}^2$.
- Compute explicit cycles $\gamma_1, \ldots, \gamma_{2g}$ on C whose cycles form a basis for homology.



- Numerically integrate the forms against the cycles.

Implementations: Seppäla *et al.* (1994), Deconinck *et al.* (2001), Swierczewski *et al.* (2016), Bruin *et al.* (2018),...

Part II

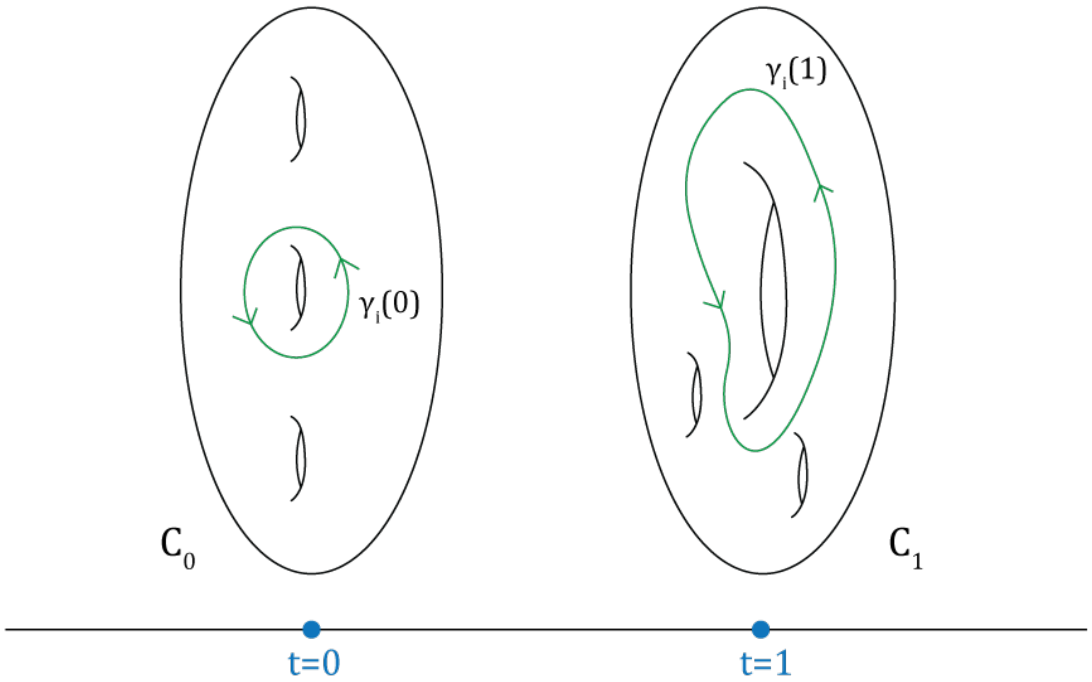
Deforming periods

Deforming loops

Suppose $C_0 = Z(f_0), C_1 = Z(f_1) \subset \mathbb{P}^2$ are two curves of degree d with C_0 smooth. Consider the family of plane curves

$$C_t : (1-t)f_0 + tf_1, \quad t \in \mathbb{C}.$$

Pick a basis $\gamma_1, \ldots, \gamma_{2g} \in \mathrm{H}_1(C_0, \mathbb{Z})$ and continue this basis along the family. This gives a basis $\gamma_1(t), \ldots, \gamma_{2g}(t) \in \mathrm{H}_1(C_t, \mathbb{Z})$ on each smooth curve.



Deforming holomorphic forms

Consider now an affine chart $\mathbb{A}^2 \subset \mathbb{P}^2$ with coordinates x, y. Pick a basis for the space of polynomials of degree not greater than $d-3$, say

$$p_1, \ldots, p_g \in \mathbb{C}[x,y]_{\leq d-3}.$$

We define the corresponding one forms

$$\omega_i(t) := p_i(x,y) \frac{\mathrm{d}x}{\partial_y f_t}\bigg|_{C_t}, \quad i = 1, \ldots, g.$$

Fact

Whenever C_t is smooth, these forms are holomorphic on C_t and form a basis for $\mathrm{H}^0(C_t, \omega_{C_t})$.

Remark

In fact, there is a way to perform this construction without appealing to affine charts, or choice of coordinates, via *Griffiths residues*.

Deforming the period matrix

Having fixed a "moving" basis for homology

$$\gamma_1(t), \dots, \gamma_{2g}(t) \in \mathrm{H}_1(C_t, \mathbb{Z})$$

and a "moving" basis for holomorphic forms

$$\omega_1(t), \dots, \omega_g(t) \in \mathrm{H}^0(C_t, \omega_{C_t})$$

we pair the two in order to define the family of period matrices:

$$\mathcal{P}(t) = \left(\int_{\gamma_j(t)} \omega_i(t) \right)_{\substack{i=1,\dots,g \\ j=1,\dots,2g}} .$$

Fact

The entries of $\mathcal{P}(t)$ vary holomorphically at t if C_t is smooth.

Picard–Fuchs equations

Let $\mathcal{P}_i(t)$ be the i-th row of the period matrix $\mathcal{P}(t)$:

$$\mathcal{P}_i(t) = \left(\int_{\gamma_1(t)} \omega_i(t), \dots, \int_{\gamma_{2g}(t)} \omega_i(t) \right) \in \mathbb{C}^{2g}.$$

Armed with the explicit polynomial representation of $\omega_i(t)$ we can compute the minimal differential operator $\mathcal{D}_i \in \mathbb{C}(t)[\partial_t]$ satisfying

$$\mathcal{D}_i \cdot \omega_i(t) \equiv 0 \in \mathrm{H}^1(C_t, \mathbb{C}),$$

where ∂_t is differentiation with respect to t. Therefore

$$\mathcal{D}_i \cdot \mathcal{P}_i(t) \equiv 0.$$

Remark

If we know the periods of C_0 then we can also compute initial conditions for $\mathcal{P}_i$ at $t = 0$. We now have an initial value problem on our hands, one for each $i = 1, \dots, g$.

Part III

Future directions

Transcendental curves

Symbolic methods work only if the defining polynomials have *nice* coefficients, e.g., if $f \in \mathbb{Q}[x, y, z]$. What if we have transcendental coefficients?

Example

Take $f = x^3 + y^3 + z^3 + \pi xyz$. This fits into the family:

$$f_t := x^3 + y^3 + z^3 + txyz, \quad t \in \mathbb{C}.$$

Here f_0 and f_1 are in $\mathbb{Q}[x, y, z]$ so we are fine.

Remark

Alternatively, one can approximate $f \in \mathbb{C}[x, y, z]$ by some $g \in \mathbb{Q}[i][x, y, z]$.

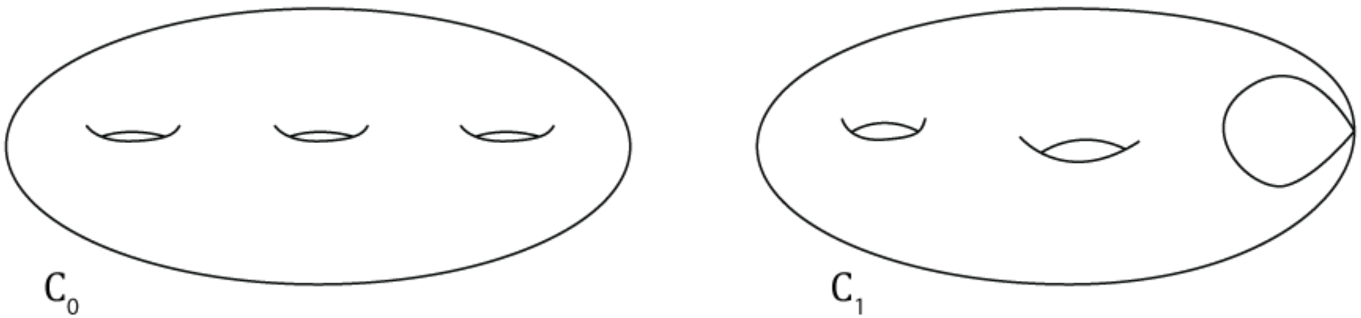
- Bound the difference between the periods of f and g.
- Take a single step in the space of periods towards the period of f, this is purely numerical.

Singular curves — take 1

Suppose our beginning curve C_0 is smooth but C_1 is not.

Example

- $C_0 : x^4 + y^4 + z^4$
- $C_1 : 2x^4 - x^3y - 2x^3z - xyz^2 - 2y^3z$

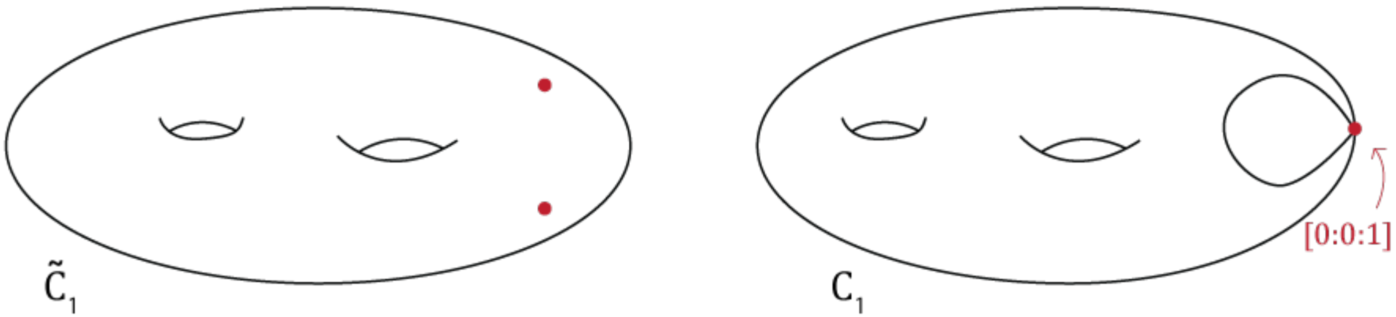


Remark

If we construct the usual family of curves $C_t : (1-t)f_0 + tf_1$ and define the corresponding family of period matrices $\mathcal{P}(t)$ then the limit, $\lim_{t\to 1} \mathcal{P}(t)$, should it exist, will be a 3×6 matrix and not 2×4.

Excess rows

In our particular example, take $(p_1, p_2, p_3) = (x, y, z)$ and define the family of holomorphic bases $\omega_1(t), \omega_2(t), \omega_3(t)$ accordingly.



In the limit, $\omega_1(1)$ and $\omega_2(1)$ will lift to a basis of holomorphic forms of $\mathrm{H}^0(\tilde{C}_1, \omega_{\tilde{C}_1})$. But $\omega_3(1)$ will lift to a *meromorphic form* with poles at the two preimages of the node.

Fact

A singularity analysis will always lead us to the right choice of polynomials which give a holomorphic basis on the resolution.

Removing excess rows

Remove the third row of $\mathcal{P}(t)$ and define the smaller period matrix:

$$\overline{\mathcal{P}}(t) = \begin{pmatrix} \mathcal{P}_1(t) \\ \mathcal{P}_2(t) \end{pmatrix}_{2\times 6}$$

Example

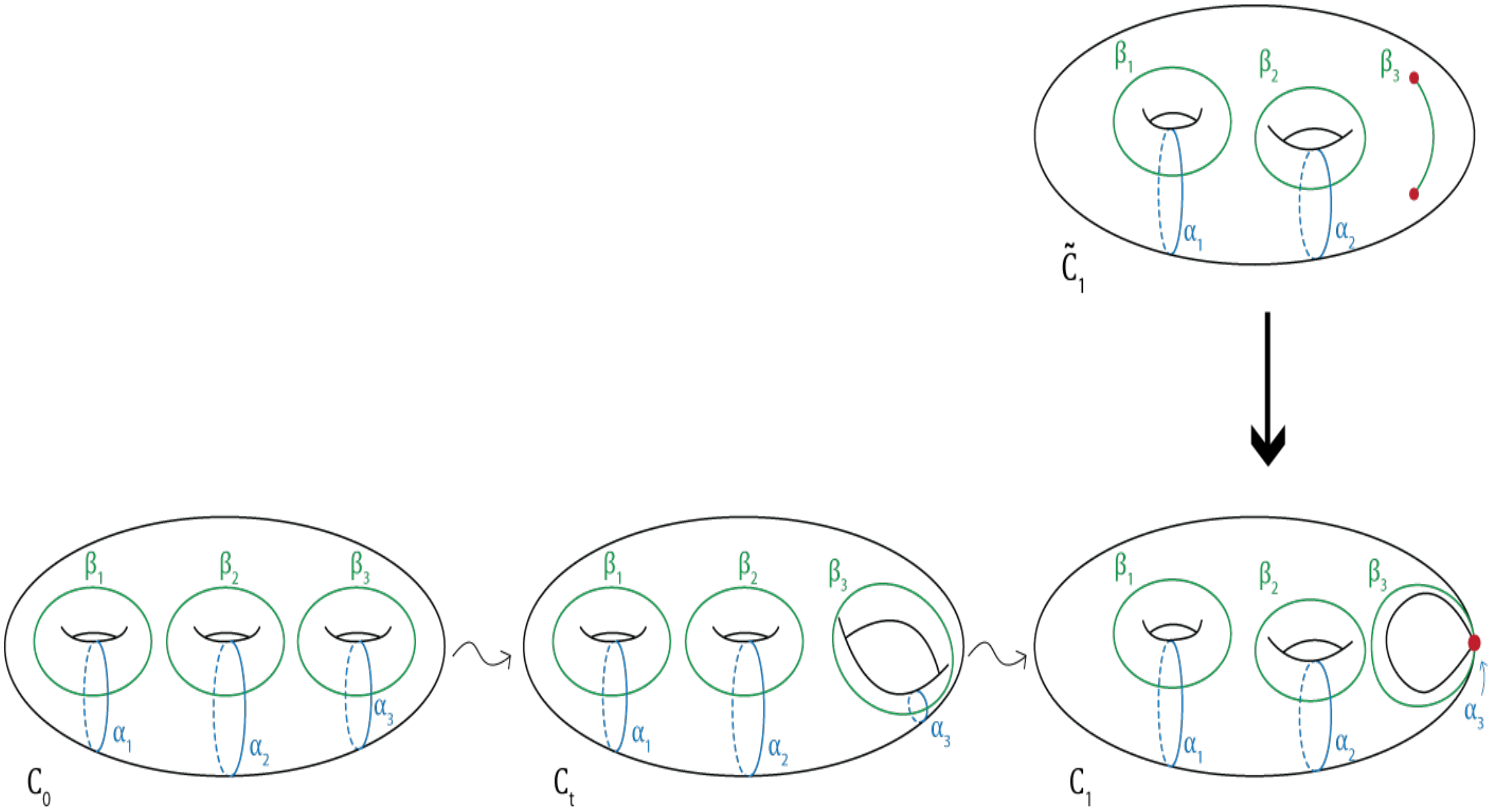
The limit of the first two rows exist:

$$\overline{\mathcal{P}}(1) := \lim_{t\to 1} \overline{\mathcal{P}}(t)$$
$$\simeq \begin{pmatrix} 1.80-0.04i & -0.69+0.57i & -1.80-0.04i & -2.22 & 0.53-1.06i & 0.95+0.04i \\ 0.48+1.11i & -0.94-0.45i & -0.48+1.11i & 0.93 & 0.57-1.32i & -0.84-1.11i \end{pmatrix}.$$

The limit of the third row does not exist:

$$\mathcal{P}_3(t) \sim \text{const.} + \begin{pmatrix} -1 & 0 & 1 & 2 & 1 & 0 \end{pmatrix} \cdot \log(1-t), \quad |1-t| \ll 0.$$

Excess columns



Removing excess columns

The loop that gets contracted to a point corresponds to a cycle

$$\alpha_3(t) = \sum_{i=1}^{6} a_i \gamma_i(t), \quad a_i \in \mathbb{Z}.$$

Since the loop gets contracted, holomorphic forms on it must evaluate to zero:

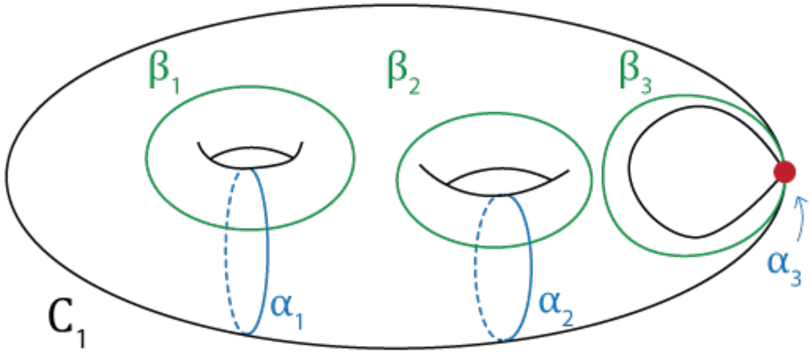
$$\overline{\mathcal{P}}(1) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix} = \lim_{t \to 1} \begin{pmatrix} \int_{\alpha_3(t)} \omega_1(t) \\ \int_{\alpha_3(t)} \omega_2(t) \end{pmatrix} = 0$$

We use LLL to compute this integer vector:

$$(a_1, a_2, a_3, a_4, a_5, a_6) = (-1, -2, 0, 0, -1, 1).$$

We can recover other loops using the intersection product, the loops we want should not intersect the vanishing cycle $\alpha_3(t)$ so we compute the orthogonal complement of $(a_1, \dots, a_6)$ in $\mathbb{Z}^6$, then remove $(a_1, \dots, a_6)$ from it:

$$\mathrm{H}_1(\tilde{C}_1, \mathbb{Z}) \simeq \alpha_3^{\perp} / \langle \alpha_3 \rangle. \tag{1}$$



Pick any four elements $\lambda_1, \dots, \lambda_4 \in \alpha_3^{\perp}$ which descend to a basis of the quotient (1), e.g., the columns of the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then the 2×4 matrix we are after is precisely:

$$\mathcal{P}_{\tilde{C}_1} = \overline{\mathcal{P}}(1) \cdot A.$$

Checking our answer

Using LLL we can compute a change of basis of cohomology and of homology to check our answer against Bruin's code:

$$\mathcal{P}_{\tilde{C}_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \mathcal{P}_{\mathsf{Bruin}} \cdot \begin{pmatrix} 0 & -1 & 1 & 2 \\ -1 & 1 & -2 & -2 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 3 & 3 \end{pmatrix}$$

Remark

We used the complicated equation

$$2x^4 - x^3y - 2x^3z - xyz^2 - 2y^3z$$

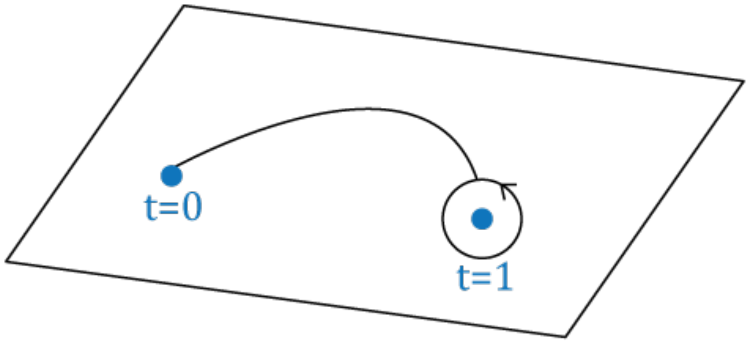
to put the preimage of the singularity in general position. The following much simpler polynomial poses a problem:

$$x^4 + xyz^2 + z^4.$$

Avoiding singularity analysis

We can do all this without "understanding" the singular fiber at all! Two ingredients are necessary:

- The limit $\lim_{t\to 1} \mathcal{P}(t)$, together with the rate of divergence.
- The monodromy operator $T : \mathrm{H}_1(C_t, \mathbb{Z}) \to \mathrm{H}_1(C_t, \mathbb{Z})$ as we go around the singular fiber once, counterclockwise.



The monodromy operator and its Jordan form:

$$T = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 & -1 \\ 1 & -1 & 0 & -1 & -1 & -1 \\ -1 & 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}, \quad S \cdot T \cdot S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It will now be sufficient to take the upper-left 2×4 block of the period matrix $\mathcal{P}(1) \cdot S^{-1}$.

Mixed Hodge structures

A piece of technology invented 50 years ago:

Theorem (Deligne,Schmidt,Steenbrink,Clemens,. . . 1970's)

The period matrix of a singular fiber in a family of projective varieties can be determined solely from the monodromy operator T and the limit period matrix.

Today seems like a good time to use it.

Thank you!