MPI MiS mini-course: Hodge theory and periods of varieties
Exercise set 3

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Numbered theorems and exercises are with reference to [1].

1. Let $d > 0$ be even and let $f(w, x, y)$ be a homogeneous polynomial of degree $d$. Let $S$ be the surface

$$S: z^2 = f(w, x, y)$$

in the weighted projective space $\mathbb{P}(1:1:1:d^2)$.

(a) Let $U_w, U_x, U_y, U_z$ be the standard “coordinate charts” of $\mathbb{P}(1:1:1:d^2)$. Prove that the only point not in any of $U_w, U_x, U_y$ is $(0:0:0:1)$.

(b) Show that $(0:0:0:1) \not\in S(\mathbb{C})$. In particular, the “chart” $S \cap U_z$ is redundant.

(c) Show that $S$ is smooth if and only if the branch curve is smooth.

(d) (Exercise 1.3.1.) Show that $S$ is connected.

2. Let $S$ be a smooth double plane. Show that $H_1(S) = 0$ implies that $H_2(S)$ is torsion-free.

3. Read [1, page 55]. Within, you will see for a double plane defined by $S_w: z^2 = f(x, y)$ on the chart $U_w$, that the differentials

$$\omega = \frac{a(x, y)dx \wedge dy}{z}$$

extend to holomorphic differentials on all of $S$.

6. Let $X$ be the blow-up of $\mathbb{P}^2$ at 6 points $\{p_1, \ldots, p_6\}$ in general position. Let $\{e_1, \ldots, e_6\}$ be the corresponding exceptional curves.

(a) Show that $\text{Pic}(X) \cong \mathbb{Z}(\ell, e_1, \ldots, e_6)$. Describe the lattice structure induced by the intersection pairing on $X$.

(b) Describe the exceptional curves on $X$ in terms of curves on $\mathbb{P}^2$ passing through $p_1, \ldots, p_6$. How many exceptional curves on $X$ are there?

(c) Determine the canonical divisor class of $X$ in terms of the basis in part (a). Show that the dual canonical bundle is very ample and compute the anti-canonical model.

(d) Let $C$ be a fixed plane cubic through the six points $p_1, \ldots, p_6$. Show that the isomorphism class of the triple $(X, C', \{e_1, \ldots, e_6\})$ is determined by $(C, \{p_1, \ldots, p_6\})$ up to linear automorphisms of $\mathbb{P}^2$, where $C'$ is the strict transform of $C$ and $e_1, \ldots, e_6$ are labeled pairwise skew exceptional curves on $X$. Note that the points are explicitly labeled as well.

Side note: We may be lead to believe that cubic surfaces have 4 moduli, coming from choosing 6 points in $\mathbb{P}^2$. This is indeed correct.
27. **Cubic surfaces!**

This exercise is based on Exercise 1.3.3. To give it a more “Non-linear algebra group” flavor, we restrict to cubic surfaces and make use of mathematical software\(^1\). Let \( S \subseteq \mathbb{P}^3 \) be a smooth cubic surface defined by \( f(x, y, z, w) = 0 \).

(a) Let \( \Omega^1_S \) be the cotangent sheaf. Compute \( \dim \mathbb{C} H^p(S, \Omega^\otimes q_S) \) for \( 0 \leq p + q \leq 4 \). (Hint: use the fact that all smooth cubic surfaces are diffeomorphic.)

(b) Compute the Hodge numbers of \( S \). Show that \( S \) has no periods.

(c) Compare the dimension counts from the previous two exercises. Try this computation for varieties in different degrees, dimensions, etc. Especially for quartic surfaces in \( \mathbb{P}^3 \).

(d) Now let

\[
X : 0 = T^3 - f(x, y, z, w) \subseteq \mathbb{P}^4
\]

Compute the Hodge numbers of \( X \).

**References**


\(^1\)I used Macaulay 2 for this exercise.