MPI MiS mini-course: Hodge theory and periods of varieties
Exercise set 4

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Numbered theorems and exercises are with reference to [1].

1. (a) Use the Mayer-Vietoris sequence to compute the cohomology of $S^1$ (the unit circle).

(b) Let $E$ be an elliptic curve, and let $U, V$ be simply connected open subsets of $E$ such that $U \cup V = E$ and $U \cap V$ consists of two cylindrical bands. Write down the Mayer-Vietoris sequence to compute the cohomology of $E$.

(c) Let $X,Y \subseteq \mathbb{A}^n$ be distinct smooth irreducible affine varieties. Show that there exists an open neighborhood $X \cap Y \subseteq U$ of $X \cup Y$ such that $X \cap Y$ is a deformation retract of $U$. Explain why the Mayer-Vietoris sequence on [1, page 63] is valid. Is there always an open neighborhood of $X \cap Y$ that admits a deformation retract onto $X \cap Y$ if $X, Y$ are path connected topological spaces?

2. Let $X: 0 = T^3 - f(x, y, z, w) \subseteq \mathbb{P}^4$ be a $\mu_3$-branched cover of a smooth cubic surface $X$. Let $\sigma: X \to X$ be a generator for the automorphism group over the branch locus $S: 0 = f(x, y, z, w) \subseteq \mathbb{P}^3$. Show that $\sigma^*$ acts on the cohomology of $X$. Show that the only $\sigma^*$-invariant classes in $H^3(X, \mathbb{R})$ are pullbacks of classes from $\mathbb{P}^3$.

3. Exercise 1.4.2.

Consider an algebraic surface $X$ with one isolated singularity at a point $p$. Let $\tilde{X}$ be a resolution of singularities, and denote the inverse image of $p$ in the resolution by $E$. The inverse image, the so-called exceptional locus, is an algebraic curve. Let us assume that it is smooth. Discuss the mixed Hodge structure on $X$ as an extension of a weight $1$ Hodge structure by one of weight $2$. Show that we get an ordinary Hodge structure if the singularity is an ordinary double point, but a genuine mixed Hodge structure if the singularity is an ordinary $m$-fold point with $m > 2$.

Warning: There is at least one typo in the section of the book which is helpful for this problem. Also, I keep getting an extension of a weight $2$ mixed hodge structure by a weight $1$ mixed hodge structure when I try to solve this. That said, I think we’re up for a challenge on Friday!

References


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Fun fact: Leopold Vietoris lived to be 110 years old, and held the record for being the oldest confirmed living Austrian.