Computing transcendental invariants of hypersurfaces via homotopy

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Overarching goal

Goal

Given the equations of a complex projective variety X, find equations for all subvarieties $Y \subset X$ with predetermined invariants (e.g. dimension, degree, Hilbert polynomial).

Conjecture (Hodge Conjecture)

If X is a smooth projective variety then $\mathrm{Alg}^k(X) \otimes \mathbb{Q} = \mathrm{Hdg}^k(X) \otimes \mathbb{Q}$.

- The term on the right is in principle easy, but the term on the left is problematic because it is very hard to write down algebraic subvarieties of a given variety.
- We will compute the term on the right for hypersurfaces and pass to the left on certain instances.
- Hodge conjecture is open already for hypersurfaces of degree six in $\mathbb{P}^5$.

Back to earth

How difficult is it to find subvarieties really?

- If $X \subset \mathbb{P}^3$ is a surface of degree $d = 1, 2, 3$ then $X \sim \mathbb{P}^2$ therefore we can describe all curves in X.
- Fun starts at $d = 4$: Symbolic methods finds lines in X, but can not even find conics!
- Quartic surfaces have cryptographic applications and finding these curves is the missing ingredient.

Challenge (For you!)

Design a numerical homotopy algorithm which takes the defining equation $f \in \mathbb{Q}[x, y, z, w]_4$ of a smooth quartic surface $X \subset \mathbb{P}^3$ and outputs:

- *(numerically) all planes containing a conic in X,*
- *certification of the computations.*

Another kind of homotopy

Advantages

Works for any smooth hypersurface $X \subset \mathbb{P}^{n+1}$ *and larger class of* $Y \subset X$.

Disadvantages

Computation time can be severe and depends on the equations of X.
Purely numerical certification would require a miracle.

Another kind of homotopy

Let $X \subset \mathbb{P}^3$ be a smooth quartic. Any curve $C \subset X$ will give rise to a homology class $[C] \in \mathrm{H}_2(X, \mathbb{Z}) \simeq \mathbb{Z}^{22}$.

- The span of curve classes is $\mathrm{Alg}^1(X) \subset \mathrm{H}_2(X, \mathbb{Z})$.
- $\mathrm{Alg}^1(X) \simeq \mathbb{Z}^\rho$ where ρ is the *Picard number of* X.

Theorem (Lefschetz (1,1)-theorem)

With ω_X a holomorphic 2-form on X, we have

$$\mathrm{Alg}^1(X) = \{\gamma \in \mathrm{H}_2(X, \mathbb{Z}) \mid \int_\gamma \omega_X = 0\}.$$

The lattice $\mathrm{Alg}^1(X)$ varies wildly with X but $\mathrm{H}_2(X, \mathbb{Z})$ is "locally constant" in families and the integrals $\int_\gamma \omega_X$ vary holomorphically as X deforms.

Period homotopy

1. Given $X = Z(f_X)$, find a simpler quartic $Y = Z(f_Y)$, e.g. the Fermat surface $f_Y = x^4 + y^4 + z^4 + w^4$, identify a basis $\gamma_1^Y, \dots, \gamma_{22}^Y \in \mathrm{H}_2(Y, \mathbb{Z})$.
2. Along the deformation $f_t = (1-t)f_Y + tf_X$, $\mathcal{X}_t = Z(f_t)$, we can (morally) define a basis $\gamma_1(t), \dots, \gamma_{22}(t) \in \mathrm{H}_2(\mathcal{X}_t, \mathbb{Z})$, such that $\gamma_i(0) = \gamma_i^Y$.
3. There is an explicit holomorphic form $\omega(t)$ on $\mathcal{X}_t = Z(f_t)$.
4. Define the period matrix $\mathcal{P}(t) = \left(\int_{\gamma_1(t)} \omega(t), \dots, \int_{\gamma_{22}(t)} \omega(t)\right)$.
5. Compute (symbolically) a differential equation $\mathcal{D} \in \mathbb{Q}(t)[\frac{\partial}{\partial t}]$ such that $\mathcal{D}\mathcal{P}(t) = 0$.
6. Compute initial conditions $\mathcal{P}(0), \mathcal{P}'(0), \mathcal{P}''(0), \dots$.
7. Numerically solve the initial value problem posed to find $\mathcal{P}(1)$.
8. At this point $\mathrm{Alg}^1(X) \simeq \ker\left(\mathcal{P}(1) \colon \mathbb{Z}^{22} \to \mathbb{C}\right)$, compute by LLL.

Finding conics

Fact

The intersection product $\mathrm{H}_2(X,\mathbb{Z})$ *is inherited from the one on* $\mathrm{H}_2(Y,\mathbb{Z})$ *and is available exactly. The induced intersection product on* $\mathrm{Alg}^1(X)$ *allows us to find curves of specified shape.*

Example

Let $X = Z(5x^4 - 4x^2zw + 8y^4 - 5z^4 + 4zw^3)$. There are 102 classes of conics in $\mathrm{Alg}^1(X) \subset \mathrm{H}_2(X,\mathbb{Z}) \simeq \mathbb{Z}^{22}$. Here is one:

$$(0, 2, -1, 1, 0, -1, 3, 0, -1, 0, -1, 0, 1, 1, 0, 2, -1, 0, 0, -1, 2, 2).$$

Cohomology classes to equations

We have thus far only used holomorphic 2-forms (i.e. $\mathrm{H}^{2,0}(X)$). Using "mixed" forms (i.e. those in $\mathrm{H}^{1,1}(X)$) reveals equations, in some limited capacity.

Let $S = \mathbb{C}[x, y, z, w]$ and $J \subset S$ the Jacobian ideal of X. Then there is an isomorphism:

$$(S/J)_4 \simeq \mathrm{H}^{1,1}(X) : p \mapsto \omega_p.$$

Given the class of a conic $C \subset X$ consider the map:

$$(S/J)_1 \times (S/J)_3 \to \mathbb{C} : (h, q) \mapsto \int_C \omega_{hq}.$$

There exists a unique linear form h annihilating every cubic q, and this form will cut out the plane containing C.

Theorem (Dan, Movasati–S.)

This method will allow you to recover the degree $\leq m$ forms in the ideal of a complete intersection $Y \subset \mathbb{P}^{n+1}$ contained in a hypersurface $X \subset \mathbb{P}^{n+1}$, for some explicit m.

Cohomology classes to equations

Let $f_X = x^4 + x^3z - xy^3 + y^4 + z^4 + w^4$, this has 56 conics. Here is one:

$$[C] = (0, 2, -1, 0, 0, -1, 2, -1, -1, 1, -2, 0, 0, 1, 1, 1, 0, 0, 0, -1, 1, 2) \in \mathbb{Z}^{22}.$$

Using the method above we find a linear form $h = x + a_1y + a_2z + a_3w$, with numerical coefficients, containing C. It turns out $a_3 = 0$ and we can find minimal relations:

$$250111a_1^{28} + 3805704a_1^{27} + \cdots + 411648a_1^2 + 8256a_1 + 64 = 0,$$

huge expression in powers of $a_1 = a_2$.

Now working over the abstract number field defined by the minimal polynomial of a_1, we can prove symbolically that X has 28 planes containing a pair of conics each. (It turns out that each pair of conics are bitangent!)

Thank you!

Code available at:
`github.com/emresertoz/PeriodSuite`