Down to characteristic p and then back up again

or, Computations with the p-adic obstruction map

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Paper:

Effective obstruction to lifting Tate classes from positive characteristic

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Code:

`github.com/edgarcosta/crystalline_obstruction`

Picard numbers of surfaces

- Let $X = Z(f) \subset \mathbb{P}^3_{\mathbb{C}}$ a smooth surface of degree d.
- Picard group $\mathrm{Pic}(X) \simeq \mathbb{Z}^\rho$, Picard number $\rho(X) = \mathrm{rk}\,\mathrm{Pic}(X)$.

$d = 1$ $\rho(X) = 1$; $X \simeq \mathbb{P}^2$

$d = 2$ $\rho(X) = 2$; $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$

$d = 3$ $\rho(X) = 7$; $X \simeq \mathrm{Bl}_{p_1,\ldots,p_6} \mathbb{P}^2$

$d = 4$ $\rho(X) \in \{1, 2, \ldots, 20\}$; each value is possible, good luck computing it!

Problem

Make computers able to calculate $\rho(X)$ from $f \in \overline{\mathbb{Q}}[x, y, z, w]_4$.

Charles (2014); Poonen, Testa, van Luijk (2014); Lairez, Sertöz (2019).
Shioda (1972-1990); van Luijk (2007); Elsenhans, Jahnel (2010's);
Hassett, Kresch, Tschinkel (2013); ...

In positive characteristic

- Let p be a prime, $\mathbb{F}_p$ field with p elements, $\overline{\mathbb{F}}_p$ algebraic closure.
- Let $X = Z(f) \subset \mathbb{P}^3_{\overline{\mathbb{F}}_p}$ a smooth surface of degree d.

$d = 1$ $\rho(X) = 1$.

$d = 2$ $\rho(X) = 2$.

$d = 3$ $\rho(X) = 7$.

$d = 4$ $\rho(X) \in \{2, 4, 6, \ldots, 22\}$. (Over $\mathbb{C}$ it was $\{1, \ldots, 20\}$.)

Problem

Make computers able to calculate $\rho(X)$ from $f \in \overline{\mathbb{F}}_p[x, y, z, w]_4$.

- Solved: Abbott, Kedlaya, Roe (2010); Costa (2015); ...
- K3 Tate conjecture: Nygaard, N., Ogus (1985); Maulik (2012); Charles (2013); Kim, Madapusi Pera (2015, 2016);

Reduction to finite characteristic

- $\frac{25}{17} \equiv 6 \mod 7$.
- $\frac{25}{17}x + 8 \in \mathbb{Q}[x]$ reduces to $6x + 1 \in \mathbb{F}_7[x]$.
- $f \in \mathbb{Q}[x, y, z, w]$ reduces to $f \mod p \in \mathbb{F}_p[x, y, z, w]$.
- $X = Z(f) \in \mathbb{P}^3_{\mathbb{Q}}$ reduces to $X_p = Z(f \mod p) \subset \mathbb{P}^3_{\mathbb{F}_p}$.

Theorem

If X and X_p are smooth then $\rho(X) \leq \rho(X_p)$.

- $X_{p^2} = Z(f \mod p^2) \subset \mathbb{P}^3_{\mathbb{Z}/(p^2)}$.

Warning

Careful:

- $f = xy - 2z^2 \in \mathbb{Q}[x,y,z]$, $f \mod 2 = xy$.
- $Z(xy) \subset \mathbb{P}^2_{\mathbb{F}_2}$ contains all points but $\{[1:1:0],[1:1:1]\}$.
- Field $\mathbb{F}_4 \simeq \mathbb{F}_2[x]/(x^2+x+1)$ versus the ring $\mathbb{Z}/(4)$.

Lifting back up, gradually

- $-1 \equiv 1 \mod 2$,
- $-1 \equiv 3 \mod 4$,
- $-1 \equiv 7 \mod 8$,
- $-1 = \frac{1}{1-2} = 1 + 2 + 4 + 8 + \ldots$ in $\mathbb{Z}_2 = \varprojlim_n \mathbb{Z}/(2^n)$.

- $\mathbb{Z} \subset \mathbb{Z}_p$, $\mathbb{Q} \subset \mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$.
- $\mathbb{Q}[x, y, z, w] \subset \mathbb{Q}_p[x, y, z, w]$.

Theorem

If $X = Z(f) \subset \mathbb{P}^3_{\mathbb{Q}}$ and $X_{\mathbb{Q}_p} = Z(f) \subset \mathbb{P}^3_{\mathbb{Q}_p}$ then $\rho(X) = \rho(X_{\mathbb{Q}_p})$.

- Clean denominators of f to get $\mathcal{X} = Z(f) \subset \mathbb{P}^3_{\mathbb{Z}_p}$.
- $\mathcal{X}$ bundles $X_{\mathbb{Q}_p}$ and $X_p, X_{p^2}, X_{p^3}, \ldots$

Cohomology theories

- Let all curve classes in X be defined over $\mathbb{Q}$.
- $\mathrm{Pic}(X) \hookrightarrow \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}) \simeq \mathbb{Q}^{22}$.
- $\mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}) = \mathrm{F}^0(X) \supset \mathrm{F}^1(X) \supset \mathrm{F}^2(X)$, Hodge filtration.
- $\mathrm{Pic}(X) \hookrightarrow \mathrm{F}^1(X) \simeq \mathbb{Q}^{21}$.

- Let all curve classes in X_p be defined over $\mathbb{F}_p$.
- $\mathrm{Pic}(X_p) \simeq \mathbb{Z}^{\rho(X_p)}$ *can not* inject into $\mathrm{H}^2_{\mathrm{dR}}(X_p/\mathbb{F}_p) \simeq \mathbb{F}_p^{22}$.
- $\mathrm{Pic}(X_p) \hookrightarrow \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ **!!**; (Berthelot 1974)

Theorem (Berthelot, Ogus 1978; Raynauld 1979)

$\mathrm{Pic}(X) = \mathrm{Pic}(X_p) \cap \mathrm{F}^1(X)_{\mathbb{Q}_p}$

Approximate the p-adic span of $\rho(X_p)$

- Frobenius: $\overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p : u \mapsto u^p$.
- Induced action $\mathrm{Frob}_p \colon \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p) \to \mathrm{H}^2_{\mathrm{dR}}(X/\mathbb{Q}_p)$.

Theorem (Tate conjecture on K3s; Nygaard, Ogus; Maulik; Charles; Kim, Madapusi Pera)

$\mathrm{Pic}(X_p) \otimes_{\mathbb{Z}} \mathbb{Q}_p = \ker(\mathrm{Frob}_p - p \cdot \mathrm{id})$

- Approximate Frob_p (Abbott, Kedlaya, Roe 2010; Costa 2015; Lauder 2011; Pancratz, Tuitman 2015)
- Compute approximate eigenvectors, done.

Remark

The map $\mathrm{H}^2 \to \mathrm{H}^2/\mathrm{F}^1$ is a coordinate projection!

- Take $f = y^4 - x^3 z + yz^3 + zw^3 + w^4$,
- $X = Z(f) \subset \mathbb{P}^3_{\mathbb{C}}$.

```
sage: crystalline_obstruction(f, p=89, precision=3)

(4,
{'dim Li': [1, 0, 3, 0], 'dim Ti': [1, 1, 4, 4],
'factors': [(t - 1, 1), (t + 1, 1), (t - 1, 4), (t^4 + 1, 1)],
'rank T(X_Fpbar)': 10})
```

- $\rho(X_{89}) = 10$
- $\rho(X) \leq 4$
- In fact, $\rho(X) \geq 4$ because there are four lines in $z = 0$.

```
sage: crystalline_obstruction(f, p=31, precision=5)

(4,
{'dim Li': [1, 1, 2], 'dim Ti': [1, 1, 2],
'factors': [(t - 1, 1), (t - 1, 1), (t + 1, 2)],
'rank T(X_Fpbar)': 4})
```

- $\rho(X_{31}) = 4$, none of the cycles seem to be obstructed.
- $\rho(X) \leq 4$, but with conviction.

Thank you!

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