Separating periods of quartic surfaces

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It is impossible to compute with real (or complex) numbers.
Work around: Consider only the numbers that arise "naturally".

Definition (Kontsevich–Zagier Periods)

$P_{\mathbb{R}} = \{$Integrals of algebraic functions over semi-algebraic domains$\}$
$P_{\mathbb{C}} = P_{\mathbb{R}} + iP_{\mathbb{R}}$

$$\pi = 2\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$$

For $a, b \in P_{\mathbb{C}}$ check equality $a = b$:

- Define a simple height function $H\colon P_{\mathbb{R}} \to \mathbb{Z}$.
- There *exists* (!)

$$\varepsilon(h) = \min\{|x-y| \mid x \neq y \in P_{\mathbb{C}},\ H(x), H(y) < h\}.$$

- Compute $a - b$ to sufficiently many digits.

Let $X/\overline{\mathbb{Q}}$ be a smooth projective variety. Taking $\gamma \in \mathrm{H}_k(X(\mathbb{C}), \mathbb{Z})$ and $\omega \in \mathrm{H}^k_{\mathrm{dR}}(X/\overline{\mathbb{Q}})$ we have

$$\int_\gamma \omega \in P_{\mathbb{C}}.$$

Period Conjecture (Kontsevich–Zagier, Grothendieck)

Loosely speaking, $\forall a, b \in P_{\mathbb{C}}$, $a = b$ iff there is a cycle in a variety over $\overline{\mathbb{Q}}$ responsible for this relation.

Our motivation:

- We can approximate periods of hypersurfaces.
- We can guess the Hodge cycles.
- We want to prove our guesses are correct, i.e., to prove a period that is approximately zero is exactly zero.

$X = Z(f) \subset \mathbb{P}^3_{\mathbb{C}}$, smooth quartic with $f \in \mathbb{Z}[x,y,z,w]_4$,

$$\omega_f = \operatorname{res} \frac{\mathrm{vol}_{\mathbb{P}^3}}{f} \in \mathrm{H}^{2,0}(X/\mathbb{Q}).$$

$$\text{Periods of } X = \left\{ \int_\gamma \omega_f \;\middle|\; \gamma \in \mathrm{H}_2(X(\mathbb{C}), \mathbb{Z}) \right\}.$$

Theorem (Lefschetz (1,1)-theorem)

$\int_\gamma \omega_f = 0 \iff \gamma$ *is algebraic, i.e.,* $\gamma = [C_1] - [C_2]$ *for* $C_1, C_2 \subset X$.

- Let $\gamma_1, \ldots, \gamma_{22} \in \mathrm{H}_2(X(\mathbb{C}), \mathbb{Z}) \simeq \mathbb{Z}^{22}$ be a basis.
- Write $\mathcal{P}_f = \left(\int_{\gamma_1} \omega_f, \ldots, \int_{\gamma_{22}} \omega_f \right)$.
- Then $\mathrm{Pic}(X) \simeq \ker \left(\mathcal{P}_f : \mathbb{Z}^{22} \to \mathbb{C} : a \mapsto \mathcal{P}_f \cdot a \right)$.
- Suppose you find $a \in \mathbb{Z}^{22}$ such that $|\mathcal{P}_f \cdot a| \sim 0$.
- How do you prove if $\mathcal{P}_f \cdot a = 0$?

$$\begin{array}{lll} \alpha \in \overline{\mathbb{Q}}: & (p_\alpha \in \mathbb{Z}[t],\ r \in \mathbb{Q}[i]), & (\deg(p_\alpha),\ \|p_\alpha\|) \\ \int_\gamma \omega_f: & (f, \gamma), & (\Delta_\gamma, \|f\|) \end{array}$$

$$\Delta_\gamma := (\gamma \cdot h_X)^2 - 4\gamma^2.$$

Definition (Noether–Lefschetz locus)

$$\mathrm{NL}_\Delta = Z(p_\Delta) \subset |\mathcal{O}_{\mathbb{P}^3}(4)|, \quad p_\Delta \in \mathbb{Z}[x_0, \dots, x_{34}]_{d_\Delta},$$

$$\mathrm{NL}_\Delta = \{\text{quartics with an algebraic cycle of discriminant } \Delta\}^-.$$

Theorem 1 (Lairez, S.)

There is an effective constant $\varepsilon(\Delta_\gamma, f) > 0$ *such that*

$$\int_\gamma \omega_f = 0 \quad \textit{or} \quad \left|\int_\gamma \omega_f\right| > \varepsilon(\Delta_\gamma, f),$$

$$\varepsilon(\Delta, f)^{-1} := \left(d_\Delta! \|p_\Delta\| \left(1 + \|f\|\right)^{d_\Delta}\right) \left(4\|\mathrm{d}\mathcal{P}_f^{-1}\|\right) \left(1 + 6\frac{\|\mathbb{I}_2\|}{\mathrm{vol}(X_f)}\right).$$

Theorem (Maulik, Pandharipande 2012)

The degrees of Noether–Lefschetz loci form the Fourier coefficients of an explicit modular form of weight $21/2$ and level 8.

Corollary

$$d_\Delta = \deg \mathrm{NL}_\Delta \leq \left(2^{21} + 2^2\right)(\sqrt{\Delta} + 1)^{21}.$$

Theorem 2 (Lairez, S.)

The height of the NL_Δ has the following bound:

$$\|p_\Delta\| \leq 2^{2^{2^{21^3 \cdot \Delta^3}}}.$$

Theorem 3 (Lairez, S.)

$\sum_{n\geq 0} \left(2 \uparrow\uparrow (5^n)\right)^{-1}$ *is not a ratio of periods of a quartic surface/$\mathbb{Q}$.*

Idea of the proof of Theorem 3

Suppose the infinite sum α is a ratio of two periods on $X = Z(f)$:

$$\alpha = \frac{\int_{\gamma_1} \omega_f}{\int_{\gamma_2} \omega_f}.$$

Partial sums give a sequence $\frac{p_n}{q_n} \in \mathbb{Q}$ approximating α,

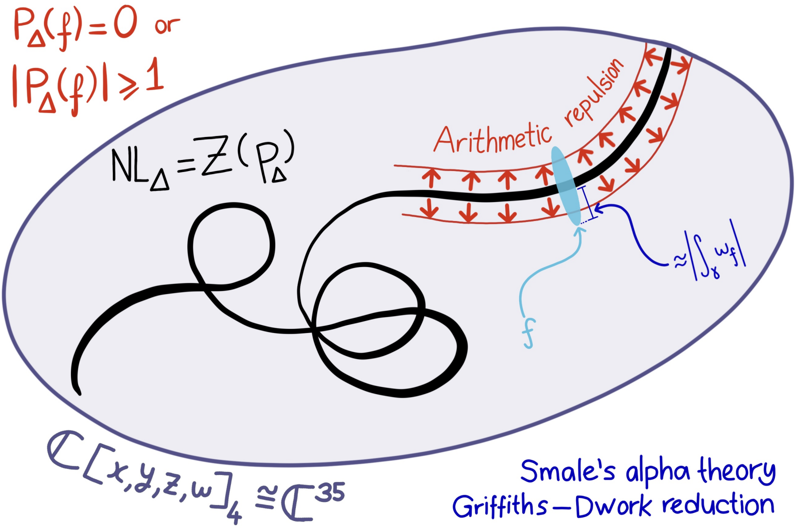
$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{c}{B_n}.$$

Here $B_n \to \infty$ much faster than q_n. Clearing denominators,

$$\left| q_n \int_{\gamma_1} \omega_f - p_n \int_{\gamma_2} \omega_f \right| < \frac{C q_n}{B_n}.$$

As $n \to \infty$, the right hand side goes to zero faster than $\varepsilon(\Delta_{q_n \gamma_1 - p_n \gamma_2}, f)$, by Theorem 2. Now apply Theorem 1 to reach a contradiction.

Idea for Theorem 1



Sketch of Theorem 2

- There is an explicit $g = O(\Delta), d = O(\sqrt{\Delta})$ such that the very general quartic in NL_Δ has a very ample curve of degree d and genus g.
- Let $p(t) = dt + 1 - g$ be the corresponding Hilbert polynomial.

$$H_{p(t)} = \{(f, I) \mid I \in \mathrm{Hilb}_{p(t)}(\mathbb{P}^3),\ f \text{ smooth quartic},\ f \in I\}^{-}.$$

- The equations for $H_{p(t)}$ are explicit enough to write down inside

$$|\mathcal{O}_{\mathbb{P}^3}(4)| \times \mathrm{Gr}(p(r), N_r) \subset |\mathcal{O}_{\mathbb{P}^3}(4)| \times \mathbb{P}^{Q_r - 1}$$

where $r = d^2 + (d-1)(d-2)/2 - g$, $N_r = \binom{r+3}{3}$ and $Q_r = \binom{N_r}{p(r)}$.

- A component of $H_{p(t)}$ projects down to NL_Δ.
- Bounding the "height" of $H_{p(t)}$, can we get to the height of NL_Δ?

Let $P = \mathbb{P}^{34} \times \mathbb{P}^{Q_r - 1}$. There is an *extended Chow ring* for products of projective spaces (D'Andrea, Krick, Sombra):

$$A^*(P) = \mathbb{R}[\eta, \theta_1, \theta_2]/(\eta^2, \theta_1^{35}, \theta_2^{Q_r}).$$

If $Y \subset P$ is defined over $\mathbb{Q}$ one can define a class:

$$[Y]_{\mathbb{Z}} \subset A^*(P).$$

The coefficients involving η record *mixed canonical heights* of Y.

- Effective cycles have positive coefficients.
- Projection with respect to a factor is monotonic, decreasing the heights.
- The class of a hypersurface is explicit in terms of its coefficients.
- Arithmetic Bézout: The class of an intersection $[X \cdot Y]_{\mathbb{Z}}$ can be bounded by a modified product of classes $[X]_{\sup}[Y]_{\sup}$.

Thank you!