Separating periods of quartic surfaces

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Real numbers are treacherous

Given $\alpha, \beta \in \mathbb{R}$ how could we check if $\alpha = \beta$?
There are three categories of problems:

Avoid false positives (Berggren–Borwein 1997)

$$\alpha = 10^{-5} \sum_{n \in \mathbb{Z}} e^{-n^2/10^{10}},$$

$$\beta = \sqrt{\pi},$$

$$0 \neq |\alpha - \beta| < 10^{-42000000000}.$$

Confirming actual equalities (Guillera 2003)

$$\sum_{n \geq 0} \frac{(\frac{1}{2})_n (\frac{1}{3})_n (\frac{2}{3})_n (\frac{1}{4})_n (\frac{3}{4})_n}{n!^5} (252n^2 + 63n + 5)(\frac{-1}{48})^n \stackrel{?}{=} \frac{48}{\pi^2}.$$

Circumventing uncountability

In any formal language, the vast majority of real numbers are inexpressible.
It is impossible to compute in the real field.

Definition (Kontsevich–Zagier Periods)

$P_{\mathbb{R}} = \{$Integrals of rational functions over semi-algebraic domains / $\mathbb{Q}\}$
$P_{\mathbb{C}} = P_{\mathbb{R}} + iP_{\mathbb{R}}$

$$\pi = 2\int_0^\infty \frac{\mathrm{d}x}{x^2+1}.$$

- Almost all known mathematical constants are periods.
- All known mathematical constants are suspected to be (exponential) periods.
- Periods appear in econometrics, biological systems, software verification, number theory, algebraic geometry, analytic combinatorics, ...

Kontsevich and Zagier (2001)

"...a large part of algebraic geometry is (in a hidden form) the study of integrals of rational functions of several variables."

Periods come from algebraic geometry

Periods of a variety

Let $X/\overline{\mathbb{Q}}$ be a smooth projective variety. Taking $\gamma \in \mathrm{H}_k(X^{\mathrm{an}}, \mathbb{Z})$ and $\omega \in \mathrm{H}^k_{\mathrm{dR}}(X/\overline{\mathbb{Q}})$ we have

$$\int_\gamma \omega \in P_{\mathbb{C}}.$$

Theorem (Folklore; Kontsevich)

Conversely, any period $\alpha \in P_{\mathbb{C}}$ comes from periods of a variety "with boundary".

Period Conjecture (Kontsevich–Zagier, Grothendieck)

Loosely speaking, $\forall a, b \in P_{\mathbb{C}}$, $a = b$ iff there is an algebraic cycle in a variety over $\overline{\mathbb{Q}}$ responsible for this relation.

Periods of quartic surfaces

$X = Z(f) \subset \mathbb{P}^3_{\mathbb{C}}$, smooth quartic with $f \in \mathbb{Z}[x,y,z,w]_4$,

$$\omega_f = \operatorname{res} \frac{\mathrm{vol}_{\mathbb{P}^3}}{f} \in \mathrm{H}^{2,0}(X/\mathbb{Q}),$$

$$\text{Periods of } X = \left\{ \int_\gamma \omega_f \;\middle|\; \gamma \in \mathrm{H}_2(X^{\mathrm{an}}, \mathbb{Z}) \right\}.$$

Theorem (Lefschetz (1,1)-theorem)

$\int_\gamma \omega_f = 0 \iff \gamma$ *is algebraic, i.e.,* $\gamma = [C_1] - [C_2]$ *for* $C_1, C_2 \subset X$.

1. Let $\gamma_1, \dots, \gamma_{22} \in \mathrm{H}_2(X^{\mathrm{an}}, \mathbb{Z}) \simeq \mathbb{Z}^{22}$ be a basis.
2. Write $\mathcal{P}_f = \left(\int_{\gamma_1} \omega_f, \dots, \int_{\gamma_{22}} \omega_f \right) \in \mathbb{C}^{22}$. (S. 2019)
3. Then $\mathrm{Pic}(X) \simeq \ker \left(\mathcal{P}_f : \mathbb{Z}^{22} \to \mathbb{C} : a \mapsto \mathcal{P}_f \cdot a \right)$. (Lairez–S. 2019)
4. Suppose you find $a \in \mathbb{Z}^{22}$ such that $|\mathcal{P}_f \cdot a| \sim 0$.
5. How do you prove if $\mathcal{P}_f \cdot a = 0$? (Lairez–S. ≈2020)

If $\gamma = [C]$ then $\gamma^2 = 2g(C) - 2$ and $\gamma \cdot h_X = \deg(C)$.

Definition (Noether–Lefschetz locus)

$$\Delta_\gamma := (\gamma \cdot h_X)^2 - 4\gamma^2.$$

$$\mathrm{NL}_\Delta = \{\text{quartics with an algebraic cycle of discriminant } \Delta\}^-,$$

$$\mathrm{NL}_\Delta = Z(p_\Delta) \subset \mathbb{P}(\mathbb{C}[x,y,z,w]_4) \simeq \mathbb{P}^{34},$$

$$p_\Delta \in \mathbb{Z}[x_0, \ldots, x_{34}]_{d_\Delta}, \quad d_\Delta := \deg(p_\Delta).$$

For a (multivariate) polynomial we write $\|\sum_u c_u x^u\| = \max_u |c_u|$.

Theorem 1 (Lairez–Sertöz 2020)

There is an effective constant $\varepsilon(\Delta_\gamma, f) > 0$ such that

$$\int_\gamma \omega_f = 0 \quad \textit{or} \quad \left|\int_\gamma \omega_f\right| > \varepsilon(\Delta_\gamma, f),$$

$$\varepsilon(\Delta, f)^{-1} := \left(d_\Delta! \|p_\Delta\| (1 + \|f\|)^{d_\Delta}\right) \left(4 \|\mathrm{d}\mathcal{P}_f^{-1}\|\right) \left(1 + 6 \frac{\|\mathbb{I}_2\|}{\mathrm{vol}(X_f)}\right).$$

Theorem (Maulik–Pandharipande 2012)

The degrees of Noether–Lefschetz loci form the Fourier coefficients of an explicit modular form of weight $21/2$ *and level* 8*.*

Corollary

$$d_\Delta = \deg \mathrm{NL}_\Delta \leq \left(2^{21} + 2^2\right)(\sqrt{\Delta} + 1)^{21}.$$

Theorem 2 (Lairez–Sertöz 2020)

The height of the NL_Δ *has the following bound:*

$$\|p_\Delta\| \leq 2^{2^{\Delta^{9/2}}}$$

Theorem 3 (Lairez–Sertöz 2020)

$\sum_{n\geq 0} (2 \uparrow\uparrow 3n)^{-1}$ *is not a ratio of periods of a quartic surface/*$\mathbb{Q}$*.*

Idea of the proof of Theorem 3

Suppose the infinite sum α is a ratio of two periods on $X = Z(f)$:

$$\alpha := \sum_{n \geq 0} (2 \uparrow\uparrow 3n)^{-1} = \frac{\int_{\gamma_1} \omega_f}{\int_{\gamma_2} \omega_f}.$$

Partial sums give a sequence $\frac{p_n}{q_n} \in \mathbb{Q}$ approximating α,

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{2}{q_{n+1}}.$$

Here q_{n+1} is much larger than q_n. Clearing denominators,

$$\left| q_n \int_{\gamma_1} \omega_f - p_n \int_{\gamma_2} \omega_f \right| = \left| \int_{q_n \gamma_1 - p_n \gamma_2} \omega_f \right| < \frac{C q_n}{q_{n+1}}.$$

As $n \to \infty$, the right hand side goes to zero faster than $\varepsilon(\Delta_{q_n \gamma_1 - p_n \gamma_2}, f)$, by Theorem 2. Now apply Theorem 1 to reach a contradiction.

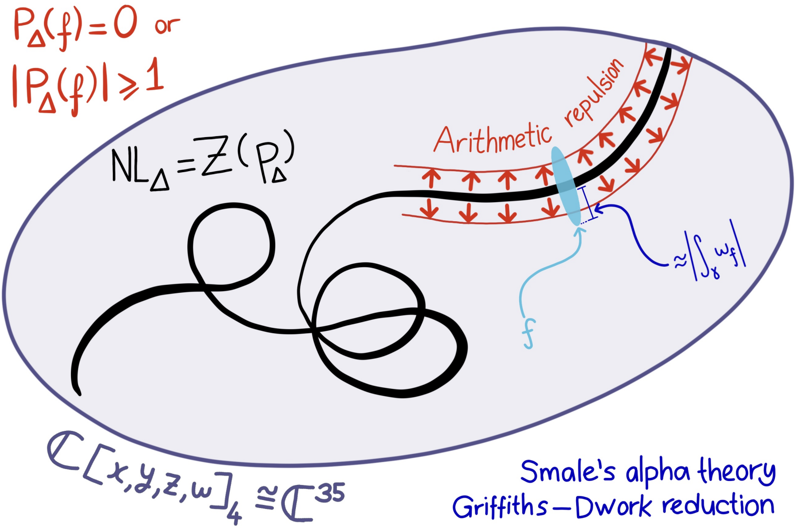
Sketch of Theorem 2

Idea

- Find something whose equations you can write down and maps onto the NL_Δ.
- Compare their "heights".

Execution

- A Hilbert scheme dominates NL_Δ.
- Hilbert schemes are famously effective in nature — although horribly complicated as a geometric object.
- We use the theory of "effective Nullstellensatz" due to D'Andrea, Krick, Sombra (2012) to control the heights as we
  - break apart the components of the Hilbert scheme,
  - map it down to NL_Δ.


$P_\Delta(f)=0$ or
$|P_\Delta(f)|\geq 1$
$NL_\Delta = Z(P_\Delta)$
Arithmetic repulsion
$\approx |\int_\gamma \omega_f|$
f
$\mathbb{C}[x,y,z,w]_4 \cong \mathbb{C}^{35}$
Smale's alpha theory
Griffiths–Dwork reduction

Thank you!